LOWPASS/BANDPASS SIGNAL RECONSTRUCTION AND DIGITAL FILTERING FROM NONUNIFORM SAMPLES

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ABSTRACT

This paper considers the problem of non uniform sampling in the case of finite energy functions and random processes, not necessarily approaching to zero as time goes to infinity. The proposed method allows to perform exact signal reconstruction, spectral estimation or linear filtering directly from the non-uniform samples. The method can be applied to either lowpass, or bandpass signals.

***Index Terms*—** Periodic nonuniform sampling, Sampling theory, Signal reconstruction, Spectral estimation, Nonuniform filtering

1. INTRODUCTION

The problem considered in this paper is the reconstruction of a stationary random process sampled at non periodic but known time instants, extending previous results concerning non uniform sampling for finite energy functions. It is also possible, without additional complexity, to derive in the same time the Fourier Transform (and spectrum) of the signal and have access to filtered versions of the process. Such a situation (non periodic and known time instants) arises in lots of applications such as:

- Extra-solar planet detection with observation of a sinusoidal behavior of the radial velocity or the brilliance of an observed star, due to its interaction with the planet to be detected. Measurements are performed only when the star is visible.
- Use of interferometers composed of a set of small size mirrors. Manufacturing defects lead to non uniform spatial distribution of the mirrors. Nevertheless, their exact position can be measured with high accuracy and can be assumed to be perfectly known.

Original contributions of this work are the following ones:

- Exact reconstruction formulas are derived.
- Ability to perform spectral estimation and linear digital filtering directly from non uniform samples.
- This can be applied to any realization of a general-type filtered random noise.
- Direct applicability to bandpass processes sampled largely below Nyquist rate, provided the frequency band is approximately known, without need of prior spectral translation.

In the context of an irregular sampling when the sampling instants are not regularly spaced but assumed to be known without error, a lot of approximate reconstruction formulas exist in the literature as polynomials or spline interpolation methods (for example [1], [2], [3]). More recent methods include papers by Selva [4] and Eldar [5] for multiband signals or Aldroubi (compressed sampling [6], data smoothing and interpolation by cubic splines [7]) or Oppenheim [8] for sinc reconstruction of bandpass signal using digital filtering. Exact formulas are difficult to find in the literature even if conditions ensuring that a band-limited signal can be reconstructed exactly from infinite irregular sampling exist [9]. In this paper, exact reconstruction formulas are derived. Section 2 presents interpolation formulas (proofs of formulas are developed) and simulations are carried out in Section 3, demonstrating the accuracy of reconstruction. Section 4 concludes the paper.

2. RECONSTRUCTION FORMULAS

Let us consider the case of a real or complex stationary processes $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ with power spectral density $s(\omega)$:

$$\mathrm{E}\left[Z(t)\, Z^*(t-\tau)\right] = \int_{-\infty}^{\infty} e^{i\omega\tau} s(\omega)\, d\omega. \tag{1}$$

where E[.] stands for the mathematical expectation and the superscript * for the complex conjugate.

2.1. General derivations for bandpass process

The random process $\mathbf{Z}$ is assumed to be bandpass. Autocorrelation function and power spectrum are derived after linear coordinate changes leading to two dimensionless frequency bands of length π

$$\mathrm{E}\left[Z(t)\, Z^*(t-\tau)\right] = \int_{-\pi-\alpha}^{-\alpha} + \int_{\alpha}^{\pi+\alpha} s(\omega)\, e^{i\omega t} d\omega, \alpha > 0. \tag{2}$$

The limit case $\alpha = 0$ corresponds to a one-piece spectrum around the origin (the process $\mathbf{Z}$ is then said to be baseband).

The method developed here has similarities with the PNS2L (Periodic Nonuniform Sampling of order $2L$) [10], [11], [12], with sampling instants chosen as in (3).

$$t_{mn} = \theta_m + 2nL, 0 < |m| \leq L, n \in \mathbb{Z} \quad (3)$$

where $\mathbb{Z}$ is the set of positive and negative integers. The main difference with previous papers on PNS2L is that only observations (samples $Z(\theta_m)$) are used in computations. As in classical reconstruction theory (sinc-based formulas), the truncation ($n > 0$) has less effect when L is large and when the rate of the θ_m is smaller than 1 (the Landau rate): $\lim_{m\to\infty} \theta_m/m = \theta < 1$.
Given a bandpass filter $G_1(\omega)$ seen as a regular function defined on $\left[\alpha, \alpha + \frac{\pi}{L}\right]$ (0 elsewhere), its shifted version can be written as

$$G_k(\omega) = \begin{cases} G_1\left(\omega - (k-1)\frac{\pi}{L}\right), k > 0 \\ G_1\left(-\omega + (k+1)\frac{\pi}{L}\right), k < 0. \end{cases} \quad (4)$$

$G_k(\omega)$ is shifted in intervals

$$\Delta_k = \begin{cases} \left(\alpha + (k-1)\frac{\pi}{L}, \alpha + k\frac{\pi}{L}\right), k > 0 \\ \left(-\alpha + k\frac{\pi}{L}, -\alpha + (k+1)\frac{\pi}{L}\right), k < 0. \end{cases} \quad (5)$$

Two processes $\mathbf{U}_k$ and $\mathbf{Z}_k$ are defined as outputs of linear invariant filters (LIF) with input $\mathbf{Z}$ and complex gains $I_{\Delta_k}(\omega)$ and $G_k(\omega)$. I_{Δ_k} is an ideal bandpass filter (indicator function of the set Δ_k).

$$\begin{cases} U_k(t) = \int_{-\infty}^{\infty} g_k(u) Z(t-u)\, du \\ G_k(\omega) = \int_{-\infty}^{\infty} g_k(u) e^{-i\omega u} du \end{cases} \quad (6)$$

$$Z_k(t) = \int_{-\infty}^{\infty} f_k(u) Z(t-u)\, du$$
$$f_k(u) = \begin{cases} e^{i\left(\alpha + \frac{k\pi}{L}\right)} \frac{e^{iu\pi/L} - 1}{2i\pi u}, k > 0 \\ e^{i\left(-\alpha + \frac{k\pi}{L}\right)} \frac{1 - e^{iu\pi/L}}{2i\pi u}, k < 0. \end{cases} \quad (7)$$

The last equality corresponds to the impulse response of an ideal bandpass filter,

$$f_k(u) = \frac{1}{2\pi} \int_{\Delta_k} e^{i\omega u} d\omega. \quad (8)$$

We are looking for formulas giving $U_k(t)$ and $Z_k(t)$ expressed as a function of the samples $Z(\theta_m + 2nL)$, so that the process can be easily reconstructed

$$Z(t) = \sum_{0<|k|\leq l} Z_k(t). \quad (9)$$

Developing the product $G_k(\omega) e^{i\omega t}$ in Fourier series on Δ_k (interval of length π/L) leads to (10) and to (11), which are valid $\forall c \in \mathbb{R}$:

$$\begin{aligned} G_k(\omega) e^{i\omega t} &= \textstyle\sum_{n\in\mathbb{Z}} a_{kn}(t) e^{2inL\omega}, \omega \in \Delta_k. \\ a_{kn}(t) &= \frac{L}{\pi} \int_{\Delta_k} G_k(\omega) e^{i\omega(t-2nL)} d\omega \end{aligned} \quad (10)$$

$$G_k(\omega) e^{i\omega t} = \sum_{n\in\mathbb{Z}} a_{kn}(t-c) e^{i\omega(c+2nL)}, \omega \in \Delta_k. \quad (11)$$

(4) and (5) then lead to (12) and equivalently to (13).

$$a_{kn}(t) = \begin{cases} \frac{L}{\pi} e^{i(k-1)\pi t/L} \int_{\Delta_1} G_1(\omega) e^{i\omega(t-2nL)} d\omega, k > 0 \\ \frac{L}{\pi} e^{i(k+1)\pi t/L} \int_{\Delta_1} G_1(\omega) e^{-i\omega(t-2nL)} d\omega, k < 0. \end{cases} \quad (12)$$

$$a_{kn}(t) = \begin{cases} b_k^+(t) c_n^+(t), k > 0 \\ b_k^-(t) c_n^-(t), k < 0 \end{cases}$$
$$b_k^+(t) = e^{it(\alpha + (k-1/2)\pi/L)}$$
$$b_k^-(t) = e^{it(-\alpha + (k+1/2)\pi/L)}$$
$$c_n^+(t) = \frac{(-1)^n L e^{-2i\alpha nL}}{\pi} \int_{-\pi/2L}^{\pi/2L} G_1\left(\omega + \alpha + \frac{\pi}{2L}\right) e^{i\omega(t-2nL)} d\omega$$
$$c_n^-(t) = \frac{(-1)^n L e^{2i\alpha nL}}{\pi} \int_{-\pi/2L}^{\pi/2L} G_1\left(\omega + \alpha + \frac{\pi}{2L}\right) e^{-i\omega(t-2nL)} d\omega \quad (13)$$

Choosing $c = \theta_m$ in (11) then allows to get (14).

$$\begin{cases} U_k(t)/b_k^+(t) = \sum_{n\in\mathbb{Z}} c_n^+(t-\theta_m) Z_k(\theta_m + 2nL), k > 0 \\ U_k(t)/b_k^-(t) = \sum_{n\in\mathbb{Z}} c_n^-(t-\theta_m) Z_k(\theta_m + 2nL), k < 0 \end{cases} \quad (14)$$

for all indices k, m that can take $2L$ values. In the last equations, the θ_m define the sampling instants, the $a_{kn}(t)$ are given by the knowledge of the function $G_1(\omega)$ (frequency pattern) and frequency band defined by α.
The samples $Z(\theta_m + 2nL)$ are known only for $n = 0$. For the $Z(\theta_m + 2nL)$ to appear from (14), parameter α and function $G_1(\omega)$ have to obey particular conditions. Let assume that

$$\alpha \in \frac{\pi}{2L}\mathbb{Z} \text{ and } G_1\left(\omega + \alpha + \frac{\pi}{2L}\right) \text{ is even.} \quad (15)$$

The latter condition means that $G_1(\omega)$ is symmetric with respect to the axis $\omega = \alpha + \frac{\pi}{2L}$. When (15) is true, formulas (14) can be summed up with respect to the index k, because $c_n^+(t) = c_n^-(t) = c_n(t)$. Then

$$\sum_{0<|k|\leq L} \frac{U_k(t)}{b_k(t-\theta_m)} = \sum_{n\in\mathbb{Z}} c_n(t-\theta_m) Z(\theta_m + 2nL) \quad (16)$$

$$b_k(t) = \begin{cases} b_k^+(t), k > 0 \\ b_k^-(t), k < 0. \end{cases} \quad (17)$$

Then $c_n^+ = c_n^- = c_n$ and

$$c_n(t) = \frac{(-1)^{n'} L}{\pi} \int_{-\pi/2L}^{\pi/2L} G_1\left(\omega + \alpha + \frac{\pi}{2L}\right) \cos\left[\omega(t - 2L)\right] d\omega \quad (18)$$

with

$$n' = n\left(1 + \frac{2\alpha L}{\pi}\right). \quad (19)$$

(16) is a linear system of size $2L\text{x}2L$ where the $U_k(t)$ are the unknowns, the indices k and m running on the set of values $-L, -L+1, ..., -2, -1, 1, 2, ..., L-1, L$. The b_k, c_n, θ_m are known. If system (16) is invertible (this property depends on the $b_k(t-\theta_m)$), the $U_k(t)$ are obtained as functions of the $Z(\theta_m + 2nL)$.

Neglecting terms for $n \neq 0$, system (16) allows to derive values of the $U_k(t)$. The following subsection show how to deduce values of $Z(t)$.

2.2. Formulas

A first formula has been derived in [13] with $G_1(\omega) = 1$ giving good results in reconstruction of both functions and spectra (compared to the Lomb-Scargle spectral estimator [14]) for band-limited deterministic functions approaching to zero as time goes to infinity:

$$\begin{gathered}\sum_{0<k\leq L} Z_k(t)\, e^{-i(t-\theta_m)(\alpha+(k-1/2)\pi/L)} + \\ \sum_{-L\leq k<0} Z_k(t)\, e^{i(t-\theta_m)(-\alpha+(k+1/2)\pi/L)} = \\ \sum_{n\in\mathbb{Z}} (-1)^{n'} \operatorname{sinc}\left[\frac{\pi}{2L}(t-\theta_m) - n\pi\right] Z(\theta_m + 2nL).\end{gathered} \tag{20}$$

The resolution of this (assumed invertible) system allows to derive the $Z_k(t)$ and then $Z(t) = \sum_{0<|k|\leq L} Z_k(t)$. Actually, the assumption that $Z(t)$ must be close to 0 as time goes to infinity is linked to the right hand side member of (20): for $n \neq 0$, unknown and neglected samples $Z(\theta_m + 2nL)$ are multiplied by a sinc function which is slowly decaying. In this paper, we show that another choice for $G_1(\omega)$ can lead to formulas with a squared sinc function (better decay) in the right hand side, allowing to extend its applicability to random processes without decay.

Let now $G_1(\omega)$ be defined on Δ_1 by

$$G_1(\omega) = \begin{cases} \frac{2L}{\pi}(\omega-\alpha), \omega \in \left(\alpha, \alpha+\frac{\pi}{2L}\right) \\ 2 - \frac{2L}{\pi}(\omega-\alpha), \in \left(\alpha+\frac{\pi}{2L}, \alpha+\frac{\pi}{L}\right). \end{cases} \tag{21}$$

With this function, the product $G_1(\omega)\, e^{i\omega t}$ is continuous on Δ_1 (its value is 0 at the bounds α and $\alpha + \frac{\pi}{L}$), which improves the convergence of its Fourier series with respect to the previous formula (20). Let define the set of shifted filters $H_k(\omega)$ (22).

$$\begin{cases} H_1(\omega) = G_1\left(\omega - \frac{\pi}{2L}\right), \omega \in \left(\alpha+\frac{\pi}{2L}, \alpha+\frac{3\pi}{2L}\right) \\ H_k(\omega) = \begin{cases} H_1\left(\omega - (k-1)\frac{\pi}{L}\right), k>0 \\ H_1\left(-\omega + (k+1)\frac{\pi}{L}\right), k<0. \end{cases} \end{cases} \tag{22}$$

The $H_k(\omega)$ are modulated versions of filters $G_k(\omega)$ with frequency shift $\pi/2L$ (half a Δ_k band). Except near the bounds of the intervals $[\alpha, \alpha+\pi]$,$[-\alpha-\pi, -\alpha]$, the sum of concatenated functions $F_k(\omega)$ and $G_k(\omega)$ is equal to the constant 1. Assuming that the spectrum $s(\omega)$ is null on intervals $\left[\alpha, \alpha+\frac{\pi}{2L}\right]$ and $\left[-\alpha-\frac{\pi}{2L}, -\alpha\right]$ leads to

$$Z(t) = \sum_{0<|k|\leq l} [U_k + V_k](t) \tag{23}$$

where $V_k(t)$ is the output of a LIF with input $Z(t)$ and complex gain $H_k(\omega)$ (defined on $\Delta_k \pm \frac{\pi}{2L}$, following the sign of k).

The FSD (Fourier Series Development) of coefficients $a_k(t)$ and $a'_k(t)$ of $G_k(\omega)\, e^{i\omega t}$ and $H_k(\omega)\, e^{i\omega t}$ are deduced from the FSD of $G_1(\omega)\, e^{i\omega t}$:

$$\begin{gathered} a_{kn}(t) = \\ \frac{(-1)^{n'}}{2} e^{i\left(\alpha+\left(k-\frac{1}{2}\right)\frac{\pi}{L}\right)(t-2nL)} \operatorname{sinc}^2\left[\frac{\pi t}{4L} - \frac{n\pi}{2}\right], k>0 \\ \frac{(-1)^{n'}}{2} e^{i\left(-\alpha+\left(k+\frac{1}{2}\right)\frac{\pi}{L}\right)(t-2nL)} \operatorname{sinc}^2\left[\frac{\pi t}{4L} - \frac{n\pi}{2}\right], k<0. \end{gathered} \tag{24}$$

$$\begin{gathered} a'_{kn}(t) = \\ \frac{(-1)^{n+n'}}{2} e^{i\left(\alpha+k\frac{\pi}{L}\right)(t-2nL)} \operatorname{sinc}^2\left[\frac{\pi t}{4L} - \frac{n\pi}{2}\right], k>0 \\ \frac{(-1)^{n+n'}}{2} e^{i\left(-\alpha+k\frac{\pi}{L}\right)(t-2nL)} \operatorname{sinc}^2\left[\frac{\pi t}{4L} - \frac{n\pi}{2}\right], k<0. \end{gathered} \tag{25}$$

Using (16) and (23), these FSDs lead to the new formulas neglecting the terms $\theta_m + 2nL$ for $n \neq 0$.

$$\begin{gathered} 2\sum_{0<k\leq L} U_k(t)\, e^{-i(t-\theta_m)(\alpha+(k-1/2)\pi/L)} + \\ 2\sum_{-L\leq k<0} U_k(t)\, e^{i(t-\theta_m)(\alpha-(k+1/2)\pi/L)} \approx \\ \operatorname{sinc}^2\left[\frac{\pi}{4L}(t-\theta_m)\right] Z(\theta_m) \end{gathered} \tag{26}$$

$$\begin{gathered} 2\sum_{0<k\leq L} V_k(t)\, e^{-i(t-\theta_m)(\alpha+k\pi/L)} + \\ 2\sum_{-L\leq k<0} V_k(t)\, e^{i(t-\theta_m)(\alpha-k\pi/L)} \approx \\ \operatorname{sinc}^2\left[\frac{\pi}{4L}(t-\theta_m)\right] Z(\theta_m). \end{gathered} \tag{27}$$

Solving linear systems (26) and (27) gives access to filtered versions of $Z(t)$ ($U_k(t)$ and $V_k(t)$) and allows reconstruction of $Z(t)$ using (23).

3. SIMULATIONS

For simulations, a time-continuous baseband random process realization $W(t)$ is first built as gaussian random noise filtered by a lowpass filter with cut-off frequency ω_c so that the total frequency band of $W(t)$ is $2\omega_c$. A realization of the process $W(t)$ is then modulated using (28) to get a bandpass process $Z_\alpha(t)$ with a two times larger total band. Note that the symmetry between the positive and negative bands is not required to apply formulas.

$$Z_\alpha(t) = W(t)\cos\left((\alpha + \frac{\pi}{2})t\right) \tag{28}$$

The case $\alpha = 0$ corresponds to the baseband case in which the spectrum of $Z_0(t)$ is in the interval $[-\pi, \pi]$ provided the frequency ω_c is chosen smaller than $\frac{\pi}{2}$. A time shifting analyzing window of length $B - A$ is considered (the process is to be reconstructed between $t = A$ and $t = B$). In simulations, $B - A$ is related to the number of samples: the condition $2L \geq B - A$ must be fulfilled to obtain a Landau rate smaller than 1.

The $2L$ sampling instants θ_m are chosen uniformly distributed over $\left[\left(m - \frac{1}{10}\right)\frac{B-A}{2L}, \left(m + \frac{1}{10}\right)\frac{B-A}{2L}\right]$, representing a non negligible jitter in the distribution of the sampling instants (10

percent). Note that even if the instants are chosen randomly for simulations, this is only a realization of the uniform distribution: the instants are then assumed to be known and we remain in the framework of deterministic sampling.
Figure of merit used in simulations is the Normalized MSE (Mean Square Error) of the reconstruction on interval $[A, B]$ defined as

$$\text{NMSE} = \left(\int_A^B \left| Z_\alpha(t) - \hat{Z}_\alpha(t) \right|^2 dt \right) / \left(\int_A^B |Z_\alpha(t)|^2 \, dt \right), \tag{29}$$

where $\hat{Z}_\alpha(t)$ is the reconstructed realization of the process $Z_\alpha(t)$.

3.1. Influence of the number of samples $2L$

In these simulations, a Landau rate slightly smaller than 1 (6/7) is assumed and $\alpha = 0$. As a consequence, $B = A + 2L\frac{6}{7}$ and the influence of L can also be interpreted as the influence of the window length $B - A$.
A process with large bandwidth is considered using $\omega_c = 0.44\pi$ and the additional condition for applying new formula $\omega_c < \frac{\pi}{2} - \frac{\pi}{2L}$ is valid only for $L \geq 9$. Figure 1 displays reconstruction NMSE using previous formulas (20) (formula 1) and new formulas (26), (27) (formula 2). Reconstruction

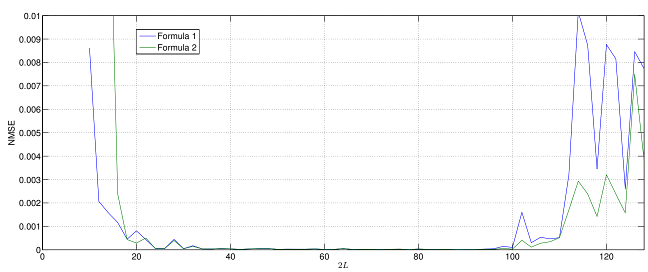

Fig. 1. Reconstruction NMSE versus the number of non uniform samples L.

NMSE with new formulas remains globally lower than NMSE obtained with previous formulas, except when $L < 9$. In this case, spectrum $s(\omega)$ is not null on intervals $\left[\alpha, \alpha + \frac{\pi}{2L}\right]$ and $\left[-\alpha - \frac{\pi}{2L}, -\alpha\right]$ and (23) is not exactly valid. Best results are achieved with small number of non uniform samples $2L$ around 70. With larger number of samples ($2L > 100$), numerical instabilities in the inversion of formulas (20), (26) and (27) can cause problems on reconstruction due to big size matrices.

3.2. Frequency shift α

The number of non uniform sample is $2L = 70$ and the process is assumed to be observed between times instants $A = 10$ and $B = 70$. As $\omega_c = 0.44\pi$, total band of the process is 1.76π. When $\alpha > 0.04\pi$, classical reconstruction cannot be performed without prior filtering and frequency shift in baseband because Shannon condition is no more fulfilled but proposed formulas can still be applied (as illustrated figure 2 for $\alpha = 0$ and figure 3 for $\alpha = 3\pi/2$), provided α is known. Using 100 Monte-Carlo runs, NMSEs obtained

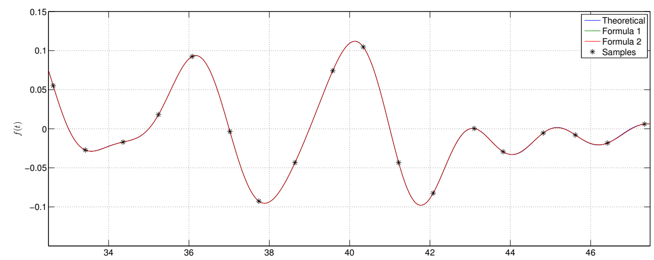

Fig. 2. Zoom on a reconstructed and theoretical realization of process $Z_0(t)$ vs time for $L = 35$ and $\alpha = 0$. The 3 curves are superimposed.

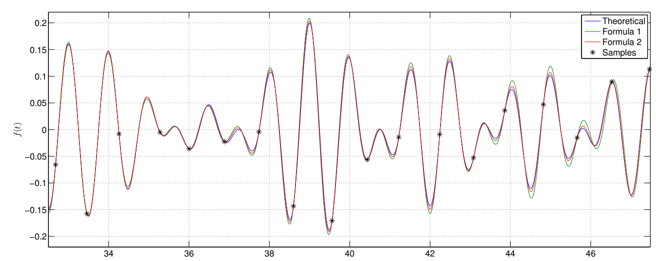

Fig. 3. Zoom on a reconstructed and theoretical realization of process $Z_\alpha(t)$ vs time for $L = 35$ and $\alpha = 3\pi/2$.

for $\alpha = 0$ with formulas 1 and 2 are respectively $1.77\text{x}10^{-5}$ and $1.72\text{x}10^{-5}$. With $\alpha = 3\pi/2$, NMSEs are respectively 0.0086 and 0.00071, showing that reconstruction can also be performed with $\alpha > 0$ for large band processes with low oversampling relatively to the Landau rate and high jitter.

4. CONCLUSION

In this paper we have given and proved new exact formulas allowing to derive at the same time estimations of reconstructed signal and corresponding power spectra when the sampling instants are not regularly spaced but assumed to be known. Formulas obtained by using PNS$2L$ (Periodic Nonuniform Sampling) scheme are shown to be valid for random processes with large band and non uniform observations with high jitter. Moreover, an original solution for performing numerical filtering directly from non uniform samples is derived from formulas, giving access at the same time to filtered versions of the process. Finally, formulas have been generalized to the case when spectral support is divided into two symmetric intervals using the general concept of Landau rate rather than Nyquist rate to highlight the real width of the spectrum.

5. REFERENCES

[1] S. Signall Y. R. Sun, “Algorithms for nonuniform bandpass sampling in radio receiver,” *Proc. ISACS*, vol. 1, pp. 25–28, 2003.

[2] P. P. Vaidyanathan, “Generalizations of the sampling theorem : seven decades after Nyquist,” *IEEE Trans on Circuits and Systems*, vol. 9, pp. 1094–1109, 2001.

[3] M. Unser P. Thevenaz, T. Blu, “Interpolation revisited,” *IEEE Trans on Med. Imag.*, vol. 7, pp. 739–758, 2000.

[4] J. Selva, “Regularized Sampling of Multiband Signals,” *Signal Processing, IEEE Transactions on*, vol. 58, no. 11, pp. 5624–5638, Nov 2010.

[5] M. Mishali and Y.C. Eldar, “From Theory to Practice: Sub-Nyquist Sampling of Sparse Wideband Analog Signals,” *Selected Topics in Signal Processing, IEEE Journal of*, vol. 4, no. 2, pp. 375–391, April 2010.

[6] A. Aldroubi and Haichao Wang, “An Adaptive and Information Theoretic Method for Compressed Sampling,” *CAMSAP, 2009 3rd IEEE International Workshop on*, pp. 193–196, Dec 2009.

[7] A. Aldroubi and H. Feichtinger, “Complete iterative reconstruction algorithms for irregularly sampled data in spline-like spaces,” *ICASSP., IEEE International Conference on*, vol. 3, pp. 1857–1860, Apr 1997.

[8] S. Maymon and A.V. Oppenheim, “Sinc Interpolation of Nonuniform Samples,” *IEEE Transactions on Signal Processing*, vol. 59, pp. 4745–4758, 2011.

[9] F. A. Marvasti, *Nonuniform sampling, theory and practice*, Kluwer Ac. Press, 2002.

[10] J. Yen, “On nonuniform sampling of bandwidth-limited signals,” *Circuit Theory, IRE Transactions on*, vol. 3, no. 4, pp. 251–257, Dec 1956.

[11] J. R. Higgins, “Some Gap Sampling Series for Multiband Signals,” *Signal Processing*, pp. 303–309, 1987.

[12] Yuan-Pei Lin and P.P. Vaidyanathan, “Periodically nonuniform sampling of bandpass signals,” *Circuits and Systems II: Analog and Digital Signal Processing, IEEE Transactions on*, vol. 45, no. 3, pp. 340–351, Mar 1998.

[13] D. Bonacci and B. Lacaze, “A new approach to spectral estimation from irregular sampling,” in *Proc. IEEE EUSIPCO-2014*, Lisbon, Portugal, September 2014.

[14] J. Scargle, “Statistical Aspects of Spectral Analysis on Unevenly Spaced Data,” *The Astronomical Journal*, vol. 263, pp. 835–853, 1982.